Lemma 3.2

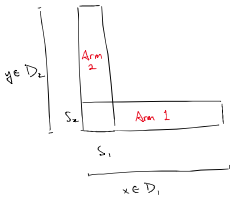
Lemma 3.2 (Extension from a cross)**.** *Let D_i be bounded connected coordinate domains and $S_i \Subset D_i$ nonempty open sets. Suppose a product solution F is given consistently on*

$$(D_1 \times S_2) \cup (S_1 \times D_2),$$

with the sum of its two L^2 norms at most M. Let $O_i \Subset D_i$ be open sets such that every L_i-solution w on D_i satisfies

$$\|w\|_{L^2(O_i)} \le C\|w\|^{\gamma_i}_{L^2(S_i)}\|w\|^{1-\gamma_i}_{L^2(D_i)}, \qquad 0<\gamma_i<1, \qquad \gamma_1+\gamma_2>1. \tag{3.1}$$

Then F extends as a product solution to $O_1 \times O_2$, agreeing with both given pieces on their intersections. On every compact subset of this product its C^m norm is at most $C_m M$. The constants are uniform when the geometric margins, ellipticity, coefficient bounds, interpolation constants and positive exponent margin are uniform. Each specified output order uses only finitely many coefficient bounds.



$f(x,y)$ on $S_1 \times D_2 \cup D_1 \times S_2$

Suppose w_j some basis of some function space on D_1 containing $f(\cdot, y)$ for $y \in S_2$

For $y \in S_2$: $f(x,y) = \sum b_j(y)\, w_j(x)$

But we know what $b_j(y)$ is outside S_2 as well!

When $x \in S_1$, $f(x,y)$ is well-defined $\forall y \in D_2$

i.e. if $b_j(y)$ can be extracted using only $w_j|_{S_1}$

then it can be extended to D_2

require: • w_j on D_1 spans enough to include f

• $w_j|_{S_1}$ keeps being orthogonal

$\Rightarrow f(\cdot, y) \in \operatorname{span}\{w_j\}$ for $y \in S_2$,

$\langle w_i, w_j\rangle_{D_1} = \delta_{ij}$; $\langle Rw_i, Rw_j\rangle_{S_1} \propto \delta_{ij}$, R restriction on S_1

$w_j(x)$ basis of $\mathcal{H}_1(D_1) := \{w \in L^2(D_1) : L_1 w = 0\}$

Consider $R : \mathcal{H}_1(D_1) \to L^2(S_1)$ restriction

$R^* : L^2(S_1) \to \mathcal{H}_1(D_1)$, $h \mapsto$ projection of 0-extension onto $\mathcal{H}_1(D_1)$

Pick w_j eigenbasis of R^*R by Spectral Thm (R^*R compact b/c R is)

then also for $i \neq j$, $\langle w_i, w_j\rangle_{S_1} = 0$

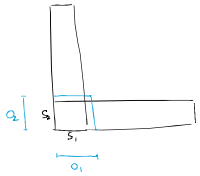
$$\Rightarrow b_j(y) = \frac{\langle f(x,y),\, w_j(x)\rangle_{S_1}}{\|w_j\|^2_{S_1}} = \mu_j^{-2}\langle f(\cdot,y), w_j\rangle_{S_1}$$

WTS: $f(x,y) = \sum_j \mu_j^{-2}\langle f(\cdot,y), w_j\rangle_{S_1} w_j$ works for $y \notin S_2$ as well!

Philosophy: One arm $(D_1 \times S_2)$ provides decomposition

The other $(S_1 \times D_2)$ provides extension blueprint

(So far, $L_2 f(x,\cdot) = 0$ has not been used!)



x-part: $w_j|_{O_1}$ y-part: $b_j(y) = \mu_j^{-2}\langle f(\cdot,y), w_j\rangle_{S_1}$

Indeed, if $O_j \approx S_j$, then $w_j|_{O_1}$ roughly controlled by $w_j|_{S_1}$

b/c w_j solution of $L_1 w = 0$ and w_j on D_1 unit size

This is actually the same for b_j : $L_2 f(x,\cdot) = 0 \Rightarrow L_2 b_j = 0$

However by on D_2 not unit size

$\|b_j\|_{D_2} = \mu_j^{-2}\|\langle f(\cdot,y), w_j\rangle_{S_1}\|_{D_2} \le \mu_j^{-2}\|f\|_{S_1\times D_2}\|w_j\|_{S_1} = \mu_j^{-1}\|f\|_{S_1\times D_2}$

$\|b_j\|_{S_2} = \|\langle f(\cdot,y), w_j\rangle_{D_1}\|_{S_2} \le \|f\|_{D_1\times S_2}\|w_j\|_{D_1} = \|f\|_{D_1\times S_2}$

For solution w of L_j , $\|w\|_{O_j} \lesssim \|w\|^{\delta_j}_{S_j}\|w\|^{1-\delta_j}_{D_j}$ with $\delta_j \approx 1$ if $O_j \approx S_j$

$$\Rightarrow \|f\|_{O_1\times O_2} = \Big\|\sum_j b_j(y)\, w_j(x)\Big\|_{O_1\times O_2}$$

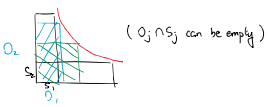
$$\le \sum_j \|b_j\|^{\delta_2}_{S_2}\|b_j\|^{1-\delta_2}_{D_2}\|w_j\|^{\delta_1}_{S_1}\|w_j\|^{1-\delta_1}_{D_1}$$

$$\lesssim \|f\|_{S_1\times D_2\cup D_1\times S_2}\sum_j \mu_j^{\delta_2+\delta_1-1}$$

Have $\mu_j = \|w_j\|_{S_1} = \|Rw_j\|_{S_1} \le C(M; S_1, D_1)\,(1+j)^{-N}$ ← (essentially $R : \mathcal{H}_1(D_1) \to L^2(S_1)$ is smoothing operator by ellipticity of L_1)

$$\Rightarrow \|f\|_{O_1\times O_2} \lesssim \|f\|_{S_1\times D_2\cup D_1\times S_2}\sum_j (1+j)^{N(1-\delta_1-\delta_2)}$$

Need: $\|f\|_{S_1\times D_2\cup D_1\times S_2} < \infty$ (away from diagonal of Green's function), $\delta_1+\delta_2 > 1$ (need to stay close to initial cross)



($O_j \cap S_j$ can be empty)

Final details: this extension is really from $D_1 \times S_2$,

need to show the extension agrees with f on $O_1\times O_2 \cap S_1\times D_2 = (S_1\cap O_1)\times O_2$

(This is trivial if O_2 connected and $S_2 \cap O_2 \neq \emptyset$, b/c both solves $L_2 w = 0$ and agree on $S_1 \times S_2$,

then UCP applies b/c $(S_1\cap O_1)\times O_2$ is connected with $S_1\times S_2$)



$D_1 \subseteq \mathbb{C}^1$

$B_a = \{z \in \mathbb{C} : |z| < a\}$

$B_{a_1} \times D_1 \cup D_1 \times B_{a_2}$

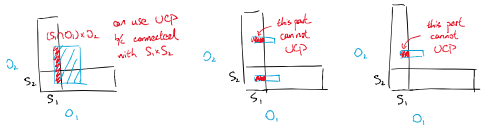
$f(z,w)$ $\bar\partial_z f = 0$ on first arm

$\bar\partial_w f = 0$ on second arm

$\|w_j\|_{O_1} \lesssim \|w_j\|^{\delta_1}_{S_1}\cdot\|w_j\|^{1-\delta_1}_{D_1}$

$= \mu_j^{\delta_1}$

(This is trivial if O_2 connected and $S_2 \cap O_2 \neq \emptyset$, b/c both solves $L_2 w = 0$ and agree on $S_1 \times S_2$, then UCP applies b/c $(S_1 \cap O_1) \times O_2$ is connected with $S_1 \times S_2$)



Suffices to show for $y \in O_2$, $f(\cdot, y)|_{S_1} \in \operatorname{span}\{Rw_j\}$

b/c then $f(\cdot, y) = \sum \langle f(\cdot, y), \frac{Rw_j}{\|Rw_j\|_{S_1}} \rangle_{S_1} \frac{Rw_j}{\|Rw_j\|_{S_1}} = \sum \mu_j^{-2} \langle f(\cdot, y), w_j \rangle_{S_1} w_j$

Let $h \in \operatorname{span}\{Rw_j\}^{\perp}$, $g_h = \langle f(\cdot, y), h \rangle_{S_1}$

Then $L_2 g_h = 0$ on D_2 and $g_h = 0$ on S_2 ($\because$ on $S_1 \times S_2$ $f(x,y) \in \operatorname{span}\{Rw_j\}$)

By UCP and connectedness of D_2, $g_h = 0$ on D_2. done

Prop 3.3

Proposition 3.3 (Product extension criterion)**.** *Let ρ be smooth near $z_0 = (x_0, y_0)$, with $\rho(z_0) = 0$ and $\mathrm{d}_x\rho, \mathrm{d}_y\rho \neq 0$. Put*

$$a_i = \frac{\nabla_i \rho}{|\nabla_i \rho|^2}, \qquad H = \operatorname{Hess} \rho.$$

Suppose that at z_0

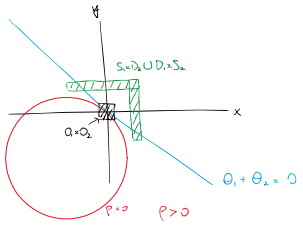
$$H((a_1, -a_2), (a_1, -a_2)) + H((v_1, v_2), (v_1, v_2)) > 0 \quad (v_i \perp a_i,\ |v_i| = |a_i|). \tag{3.7}$$

Every product solution given on $\{\rho > 0\}$ near z_0 extends to a neighborhood of z_0, agreeing on a smaller part of that side. In fact, the construction uses only two compact product sets in $\{\rho > \epsilon\}$ for some $\epsilon > 0$. The neighborhoods and estimates on smaller neighborhoods are uniform under small smooth perturbations preserving the strict hypotheses, provided the data on those compact sets are bounded.

(3.7) $\operatorname{Hess}\rho((a_1, -a_2), (a_1, -a_2)) + \operatorname{Hess}\rho((v_1, v_2), (v_1, v_2)) > 0$

gives Carleman weights $\theta_1(x)$, $\theta_2(y)$ s.t. two things hold:

① Geometrically: ρ is "more convex" than $\theta_1 \oplus \theta_2$ s.t. a cross shape exists on $\{\rho > 0\}$ (the known side) but the entire "box" extends to $\{\rho < 0\}$



$\theta_1 = x \quad \theta_2 = y$

$\rho = x - y + \frac{1}{4}(x^2 + y^2)$

$D_1 = D_2 = (-1, 1)$

$S_1 = S_2 = (0.8, 1)$

$O_1 = O_2$ nbd of 0

② Analytically: Can arrange S, O, D s.t. interpolation estimate holds

Step 1: ρ is "more convex" than direct sum of two Carleman weights

If θ_i Carleman weight and $d\theta_1(x_0) = d_x\rho(x_0, y_0)$, $d\theta_2(y_0) = d_y\rho(x_0, y_0)$, then

$$\operatorname{Hess}\theta_i(a_i, a_i) + \operatorname{Hess}\theta_i(v, v) > 0, \quad a_i = \frac{\nabla_i \rho}{|\nabla_i \rho|^2} = \frac{\nabla \theta_i}{|\nabla \rho|^2}, \quad |v| = |a_i|, \ v \perp a_i$$

Want to show:

$$(3.7) \iff \exists\, P_1, P_2 \text{ s.t. } \begin{cases} P_i(a_i, a_i) + P_i(v, v) > 0 \\ \operatorname{Hess}\rho + K\, d\rho \otimes d\rho > P_1 \oplus P_2 \text{ for some } K > 0 \end{cases}$$

$\Leftarrow$: Because $(a_1, -a_2)$, (v_1, v_2) both tangent to $\{\rho = 0\}$,

$$\operatorname{Hess}\rho((a_1, -a_2), (a_1, -a_2)) + \operatorname{Hess}\rho((v_1, v_2), (v_1, v_2))$$

$$> P_1(a_1, a_1) + P_2(a_2, a_2) + P_1(v_1, v_1) + P_2(v_2, v_2) > 0$$

only need this direction

$\Rightarrow$: Strict semidefinite feasibility problem:

Find P_1, P_2, K such that

$\tilde{\rho} = \rho + \frac{K}{2}\rho^2$

$\tilde{\rho}(x, y) > \theta_1(x) + \theta_2(y) + c(|x - x_0|^2 + |y - y_0|^2)$

only need this direction

$> P_1(a_1,a_1) + P_2(a_2,a_2) + P_1(v_1,v_1) + P_2(v_2,v_2) > 0$

$\Rightarrow$: Strict semidefinite feasibility problem:

Find P_1, P_2, K such that

$$\begin{cases} H + K\nu\otimes\nu - P_1\oplus P_2 > 0 \\ M_1 > 0,\ M_2 > 0,\ K > 0 \end{cases}$$

There are 3 positive definiteness we need to satisfy. $K>0$ is just to remove normal direction, not too important.

where $M_i := P_i + P_i(a_i,a_i)\, I\,|_{a_i^\perp}$

Positive definiteness of a matrix can be tested by positive semi-definite matrices ≥ 0

Consider the cone defined by Carleman estimate:

$$\mathcal{C} := \{ P\in \mathrm{Sym} : P(a,a) + P(v,v) > 0 \quad \forall v\perp a,\ |v| = |a| \}$$

The dual cone $\mathcal{C}^* = \{ D : P:D \geq 0 \ \forall P\in\mathcal{C} \}$

$= \mathrm{cone}\{ a\otimes a + v\otimes v : v\perp a,\ |v| = |a| \}$

$= \{ d\, a\otimes a + S : d\geq 0,\ S\geq 0 \text{ on } a^\perp,\ \mathrm{tr}\, S = d|a|^2 \}$

Let $\mathcal{C}_i$ be the cone for P_i, since $(P_1\oplus P_2):D = P_1:D_{11} + P_2:D_{22}$ where $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$

$\Rightarrow$ Dual test matrices for $P_1\oplus P_2$ are $\{D : D_{ii}\in\mathcal{C}_i^*\}$

We want $H + K\nu\otimes\nu \in \widetilde{\mathcal{C}} := \{ Q + P_1\oplus P_2 : Q>0,\ P_i\in\mathcal{C}_i \}$

$\widetilde{\mathcal{C}}^* = \{ D\geq 0 : D_{ii}\in\mathcal{C}_i^* \}$

$(H + K\nu\otimes\nu):D$

$H + K\nu\otimes\nu \in \widetilde{\mathcal{C}} \iff H:D + K\nu^T D\nu \geq 0 \quad \forall D\in\widetilde{\mathcal{C}}^*$

Obstruction: $D\in\widetilde{\mathcal{C}}^*$ s.t. $D\nu = 0$ and $H:D < 0$

If $\exists D\in\widetilde{\mathcal{C}}^*$ s.t. $D\nu = 0$, $H:D\leq 0$ and feasibility problem holds, then

$H:D + K\nu\otimes\nu : D^0 - (P_1\oplus P_2):D = H:D - (P_1\oplus P_2):D \leq H:D \leq 0$, contradiction

Thus having such a D $\Rightarrow$ Feasibility problem fails

By Convex Separation Theorem, if such $\ldots$, then feasibility problem holds

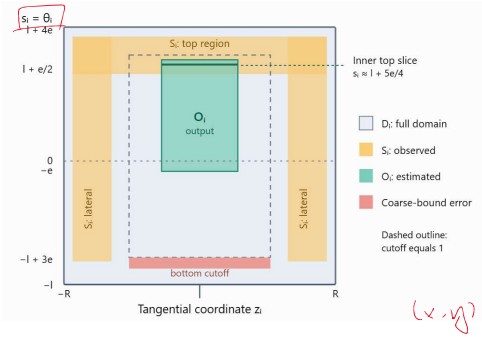
Let $D\in\widetilde{\mathcal{C}}^*$, $D\nu = 0$ (recall $\nu = dp$)

By some linear algebra computation, $D = d\,(a_1,-a_2)^{\otimes 2} + D_\perp$, $D_\perp = \sum_\alpha (u_\alpha, v_\alpha)^{\otimes 2}$. $\mathrm{tr}(D_\perp)_{ii} = d|a|^2$

However, $H:D = \sum_\alpha \left(H((a_1,-a_2),(a_1,-a_2)) + H((v_1^\alpha, v_2^\alpha),(v_1^\alpha, v_2^\alpha)) \right) > 0$ by (3.7)!

Hence such D doesn't exist, feasibility problem holds!

Step 2: P_i gives Carleman weigh θ_i s.t. $\mathrm{Hess}\,\theta_i = P_i$, use θ_i to construct cross



larger exponential weight

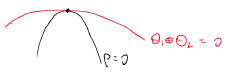
+

stronger a priori control

(in the sense that have to rely on S)

$(x,y)\in S_1\times D_2$ $\quad S_1 + S_2$

Can adjust s.t. $S_1\times D_2 \cup D_1\times S_2 \subseteq \{P>0\}$



b/c extra convexity of P provides extra room,

and S, D designed s.t. only one of x, y can be close to 0

Next need to show interpolation estimate and $\gamma_1 + \gamma_2 > 1$

Roughly: Once θ is found, Carleman estimate holds for φ slightly less pseudoconvex i.e. $\varphi = \theta - \kappa\theta^2$

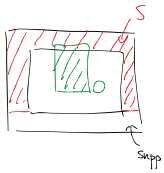
Take χ cutoff st. $\forall\chi$ supp in a rectangular annulus around O

Farkas' lemma— Let $\mathbf{A}\in\mathbb{R}^{m\times n}$ and $\mathbf{b}\in\mathbb{R}^m$. Then exactly one of the following two assertions is true:

1. There exists an $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x}\geq 0$.
2. There exists a $\mathbf{y}\in\mathbb{R}^m$ such that $\mathbf{A}^\top\mathbf{y}\geq 0$ and $\mathbf{b}^\top\mathbf{y} < 0$.

Roughly: Once Θ is found, Carleman estimate holds for φ slightly less pseudoconvex i.e. $\varphi = \Theta - \kappa\Theta^2$

Take χ cutoff st $\nabla\chi$ supp in a rectangular annulus around O



$\|e^{\tau\varphi}\chi u\| \le \|e^{\tau\varphi}L(\chi u)\| \approx \|e^{\tau\varphi}(L\chi)u\|$

$\Rightarrow \|e^{\tau\varphi}u\|_O \lesssim \|e^{\tau\varphi}u\|_S + \|e^{\tau\varphi}u\|_{\text{bottom cutoff}}$

$\Rightarrow e^{\tau\cdot\varphi(-\epsilon)}\|u\|_O \le \|e^{\tau\varphi}u\|_O \le \|e^{\tau\varphi}u\|_S + \|e^{\tau\varphi}u\|_{\text{bottom cutoff}}$

$\lesssim e^{\tau\cdot\varphi(\ell+\epsilon/3)}\|u\|_S + e^{\tau\varphi(-\ell+\epsilon/3)}\|u\|_D$

$\Rightarrow \|u\|_O \lesssim e^{\tau\cdot A}\|u\|_S + e^{\tau\cdot B}\|u\|_D$,

$A = \varphi(\ell+\epsilon/3) - \varphi(-\epsilon) > 0$, $B = \varphi(-\ell+\epsilon/3) - \varphi(-\epsilon) < 0$



$\nabla\chi = 0$

$\|u\|_D$

$S_1 \subset D_2 \subseteq$ known

Choose τ to balance two terms:

$\tau = k_S \log\|u\|_S + k_D \log\|u\|_D$

$e^{\tau A}\|u\|_S + e^{\tau B}\|u\|_D = \|u\|_S^{Ak_S+1}\cdot\|u\|_D^{Ak_D} + \|u\|_S^{Bk_S}\cdot\|u\|_D^{Bk_D+1}$

$\begin{cases} Ak_S+1 = Bk_S \\ Ak_D = Bk_D+1 \end{cases} \Rightarrow \begin{cases} k_S = \frac{1}{B-A} \\ k_D = \frac{1}{A-B} \end{cases}$

$\Rightarrow \|u\|_O \le \|u\|_S^{\frac{-B}{A-B}}\|u\|_D^{\frac{A}{A-B}}$

$\gamma_1 = \frac{-B}{A-B} = \frac{\varphi(-\epsilon)-\varphi(-\ell+\epsilon/3)}{\varphi(\ell+\epsilon/3)-\varphi(-\ell+\epsilon/3)} > \frac{1}{2}$

b/c at $\epsilon=0$ it is $\frac{0-(-\ell)+\kappa\ell^2}{\ell-\kappa\ell^2-(-\ell)+\kappa\ell^2} = \frac{1}{2}+\frac{\kappa}{2}\ell$

Now apply lemma 3.2

Pseudoconvexity / Pseudoconcavity

① Carleman Estimate

② Pseudoconvexity for scalar Laplacian Δ

③ Pseudoconvexity for $\bar\partial$

④ Pseudoconvexity in SCV / Levi form

⑤ Product Pseudoconvexity (3.7)

① $Pu = 0$



want to use $e^{\tau\phi}$ to emphasize $\phi>0$ region.

$\|e^{\tau\phi}u\| \le \|e^{\tau\phi}Pu\|$

Equivalently: $\|v\| \lesssim \|P_\tau v\|$ where $v = e^{\tau\phi}u$, $P_\tau = e^{\tau\phi}Pe^{-\tau\phi}$

if $\sigma(P) = p(x,\xi)$, then $\sigma(P_\tau) = p(x,\tau-i\tau\nabla\phi) = \mathrm{Re}\,p + i\,\mathrm{Im}\,p$

$\|P_\tau v\|^2 \approx \|\mathrm{Re}P_\tau v\|^2 + \|\mathrm{Im}P_\tau v\|^2 + 2i\langle[\mathrm{Re}P_\tau, \mathrm{Im}P_\tau]v, v\rangle$

Need $[\mathrm{Re}P_\tau, \mathrm{Im}P_\tau] > 0$ when $\mathrm{Re}P_\tau, \mathrm{Im}P_\tau$ loses control

Namely, on $\{\mathrm{Re}\,p = \mathrm{Im}\,p = 0\}$, $\{\mathrm{Re}\,p, \mathrm{Im}\,p\} > 0$

this is pseudoconvexity of ϕ w.r.t. elliptic operator P

$e^{\tau\phi}Pe^{-\tau\phi}$

$\mathrm{Re}\,p_\tau + i\,\mathrm{Re}\,p_\tau$

② Plug in $P = \Delta_g$, obtain: $\mathrm{Hess}\,\phi(\nabla\phi,\nabla\phi) + \mathrm{Hess}\phi(v,v) > 0$

$\forall\, v \perp \nabla\phi$, $|v| = |\nabla\phi|$

(Equivalently, $\frac{\nabla\phi}{|\nabla\phi|}(\log|\nabla\phi|) + \mathrm{II}(v,v) > 0$ where II s.f.f.)

Since testing with 1 normal and 1 tangential, choice of ϕ matters

Convexification: if ϕ is any ϕ of any smooth hypersurface

$\tilde\phi = \phi + \frac{K}{2}\phi^2$ for K large

$\mathrm{Hess}\,\tilde{\phi} = \mathrm{Hess}\,\phi + K\, d\phi \otimes d\phi$, $\nabla\tilde{\phi} = \nabla\phi$

Hence $\mathrm{Hess}\,\tilde{\phi}(\nabla\tilde{\phi}, \nabla\tilde{\phi}) + \mathrm{Hess}\,\tilde{\phi}(v,v)$

$= \mathrm{Hess}\,\phi(\nabla\phi, \nabla\phi) + \mathrm{Hess}\,\phi(v,v) + K|d\phi|^4$

> 0 as long as K suff large

⑤ Plug in $P = \bar{\partial}$, obtain:

$P_t = e^{t\phi}\bar{\partial}e^{-t\phi} = \bar{\partial} - \tau\bar{\partial}\phi = \frac{1}{2}[\partial_x + i\partial_y - \tau\partial_x\phi - i\tau\partial_y\phi]$

$= \underbrace{\frac{1}{2}[i\partial_y - \tau\partial_x\phi]}_{\Downarrow} + \underbrace{\frac{1}{2}[\partial_x - i\tau\partial_y\phi]}_{\Downarrow}$

$\mathrm{Rep} = -\eta - \tau\partial_x\phi \qquad \mathrm{Imp} = \xi - \tau\partial_y\phi$

$\{\mathrm{Rep}, \mathrm{Imp}\} = \left(-\partial_y + \tau\partial_x^2\phi\,\partial_\xi\right)\left(\xi - \tau\partial_y\phi\right) = \tau\Delta\phi$

Pseudoconvexity is $\Delta\phi > 0$, namely $\partial\bar{\partial}\phi > 0$, on $\{(x,y,\tau\partial_y\phi, -\tau\partial_x\phi)\}$ (so all (x,y) at $\phi = 0$)

Similarly, convexification: $\Delta(\phi + \frac{K}{2}\phi^2) = \Delta\phi + K|\nabla\phi|^2 + K\phi\Delta\phi$

$= \Delta\phi + K|\nabla\phi|^2$ at $\phi = 0$

> 0 for K large

So it also depends on normal directions, and any hypersurface works

(This proves any holomorphic function on $\mathbb{C}$ vanishing on an open set vanishes everywhere)

Now on $\mathbb{C}^n$, consider $\bar{\partial}$ where $\bar{\partial}f = \sum \partial_{\bar{z}_j} f\, d\bar{z}_j$

Then $p = (p_1, \cdots, p_n)$, so $p_t = \left(p_j(z, \zeta_j - i\tau\partial\phi)\right) = \left(i\zeta_j - \tau\partial_{\bar{z}_j}\phi\right)$

$\{\mathrm{Rep}, \mathrm{Imp}\} = \frac{1}{2i}\{p_t, p_t\} = \frac{1}{2i}\left(\{\overline{p_{j,\tau}}, p_{k,\tau}\}\right)_{j,k}$

$= \frac{1}{2i}\left(\{-i\zeta_j - \tau\partial_{\bar{z}_j}\phi, i\zeta_k - \tau\partial_{\bar{z}_k}\phi\}\right)_{j,k}$

$= \frac{1}{2i}\left(i\tau(\partial_{z_j} - i\partial_{\xi_j})\partial_{\bar{z}_k}\phi + i\tau(\partial_{z_k} + i\partial_{\xi_k})\partial_{\bar{z}_j}\phi\right)_{j,k}$

$= \tau\left(\partial_{z_j}\partial_{\bar{z}_k}\phi\right)_{j,k}$ ← this is $H_{\mathbb{C}}\phi$!

it is not appropriate to call $H_{\mathbb{C}}\phi > 0$ the pseudoconvexity for $\bar{\partial}$ on $\mathbb{C}^n$ b/c the condition is specific for scalar operators

④ Let ϕ be C^2 function on $\mathbb{C}^n$ real valued

Denote $H_{\mathbb{C}}\phi := \left(\partial_{z_j}\partial_{\bar{z}_k}\phi\right)_{j,k}$ complex Hessian

Then ϕ is plurisubharmonic if $H_{\mathbb{C}}\phi \geq 0$, strictly psh if $H_{\mathbb{C}}\phi > 0$

Define Levi form $L_\phi := H_{\mathbb{C}}\phi\big|_{T^{\mathbb{C},1,0}_\phi}$ restriction of $H_{\mathbb{C}}\phi$ on complex tangent vectors

Since Levi form only acts on tangent vectors, it is a geometric form for hypersurface, independent of choice of defining function

$\Omega \subseteq \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex if $L_\phi \geq 0$ for some bdf ϕ

strictly Levi pseudoconvex if $L_\phi > 0$

Levi pseudoconcave if $L_\phi < 0$

strictly Levi pseudoconcave if $L_\phi < 0$

with the convention $\Omega = \{\phi < 0\}$!!

The Levi problem: In $\mathbb{C}^n$, any pseudoconvex domain has inextendible holomorphic function

At a boundary point where L_ϕ has negative eigenvalue,

holomorphic functions in domain admits local extension

⑤ For Prop 3.3, similar to Levi form, (3.7) only requires tangential vectors

Note that in prop 3.3 extension is from $\{\rho > 0\}$ to $\{\rho < 0\}$

(Holomorphic extension from $\{\sigma < 0\}$ to $\{\sigma > 0\}$ requires Levi form have positive eigenvalue)

Suppose $M_1 = M_2 = \mathbb{R}^2$, $g_1 = g_2 = g_{Euc}$, $\Sigma = \{\rho = 0\}$, $\Omega_+ = \{\rho > 0\}$, $A = (a_1, -a_2) \in T\Sigma$

Define complex structure $J_1\partial_{x_1} = \partial_{y_1}$, $J_1\partial_{y_1} = -\partial_{x_1}$

Consider complex structures on $\mathbb{R}^2 \times \mathbb{R}^2$:

$J_+ = J_1 \oplus J_2$, $J_- = J_1 \oplus -J_2$

$J_\pm A = \left(J_1 a_1, \pm J_2(-a_2)\right) = \left(J_1 a_1, \mp J_2 a_2\right) \in T\Sigma$

Hence $T^{\mathbb{C},\pm}\Sigma = T\Sigma \cap J_\pm T\Sigma = \mathrm{span}_{\mathbb{R}}\{A, J_\pm A\}$

Since $|v_i| = |a_i|$, $v_i \perp a_i$, we have $v_i = \pm J_i a_i$

Possible $V = (v_1, v_2)$ are: $(J_1 a_1, J_2 a_2) = J_- A$, $(J_1 a_1, -J_2 a_2) = J_+ A$,

$(-J_1 a_1, J_2 a_2) = -J_+ A$, $(-J_1 a_1, -J_2 a_2) = -J_- A$

So (3.7) $\Leftrightarrow$ $\begin{cases} \mathrm{Hess}\rho(A, A) + \mathrm{Hess}\rho(J_+A, J_+A) > 0 \\ \mathrm{Hess}\rho(A, A) + \mathrm{Hess}\rho(J_-A, J_-A) > 0 \end{cases}$

J^+ $f(z_1, z_2)$

In $(\mathbb{R}^2, g_{Euc}, J)$, let $Z = \frac{1}{2}(A - iJA) \in T^{\mathbb{C}}\Sigma$

$\Delta_1 f = 0$ $\Delta_2 f = 0$

J^- $f(z_1, \overline{z_2})$

$$\mathcal{L}^J_\rho(Z, \overline{Z}) = \frac{1}{4}\left(\mathrm{Hess}\rho(A, A) + \mathrm{Hess}\rho(JA, JA)\right)$$

Thus (3.7) $\Leftrightarrow$ $\begin{cases} \mathcal{L}^{J_+}_\rho > 0 & \text{on } T^{\mathbb{C},+}\Sigma \\ \mathcal{L}^{J_-}_\rho > 0 & \text{on } T^{\mathbb{C},-}\Sigma \end{cases}$ pseudoconcavity of Levi form ($\{\rho > 0\}$ is known region)

Extension of product solution requires Levi extension condition for both complex structures

i.e. both $\underbrace{(x_1 + iy_1)(x_2 + iy_2)}_{\text{holomorphic for } J_+}$ and $\underbrace{(x_1 + iy_1)(x_2 - iy_2)}_{\text{holomorphic for } J_-}$ satisfy $\Delta_1 u = 0$, $\Delta_2 u = 0$

The criterion must allow extension of both J_+ and J_- holomorphic functions!

More generally, J_+ holomorphic means $f(z, \bar{z})$, J_- holomorphic means $f(\bar{z}, z)$

Locally a product solution satisfies $F(z_1, \bar{z}_1) = F_{++}(z_1, z_2) + F_{+-}(z_1, \bar{z}_2) + F_{-+}(\bar{z}_1, z_2) + F_{--}(\bar{z}_1, \bar{z}_2)$

If F real valued, then $F(z_1, \bar{z}_2) = \mathrm{Re}\, f(z_1, z_2) + \mathrm{Re}\, g(z_1, \bar{z}_2)$ by combining conjugate terms

Hence the two complex structures J_+ and J_-

The paper requires $\dim M \geq 3$, so there isn't canonical complex structure

still, let $Z_1 = a_1 + iv_1$, $Z_2 = -a_2 + iv_2$, $Z = (Z_1, Z_2)$, then

$$d\rho(Z) = 0, \quad g_1(Z_1, Z_1) = g_2(Z_2, Z_2) = 0$$

Z complex tangent vector with each component in complex characteristic set of Δ_{g_j}

(3.7) $\Leftrightarrow$ $\mathrm{Hess}\rho(Z, \bar{Z}) > 0$ $\forall$ such $Z = (Z_1, Z_2)$